Concerning Space-Time and Symmetry Groups*

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Several theorems are proved which exhibit the impossibility of constructing nontrivial products of the Lorentz group with internal symmetry groups for some physically interesting examples. It is also shown that if the Lorentz group is replaced by the Galilei group, qualitatively different results are obtained; a "linear" breakdown of internal symmetry is possible.

I. INTRODUCTION

THE existence of approximate internal symmetries within the framework of a relativistic quantum theory and, in particular, the possibility of a broken symmetry being the manifestation of a larger exact symmetry group (which subsumes the Lorentz group) has been discussed recently by several authors.[1–3] (See also papers quoted in Ref. 6.)

We might, as a first step, consider a Lie group which is a product[4] of the internal symmetry group and the Lorentz group. In this connection, McGlinn[3] has shown that if the homogeneous Lorentz transformations commute with all the generators of the internal symmetry group, then it has to reduce to a direct product for a large class of symmetry groups, larger than, but including, the class of semisimple groups. We will rederive McGlinn's result in this note in a more transparent manner which will enable us to make a precise characterization of these groups: they are groups with Abelian factor groups (and these include all solvable groups). McGlinn's result has the implication that for most internal symmetry groups of interest the Hamiltonian must commute with the internal symmetry generators so that mass splitting within the multiplets cannot be obtained within this framework.

In the following section we will derive McGlinn's result, and state and prove several related results. We shall work exclusively with the Lie algebras rather than with Lie groups.

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[1] Such attempts have been discussed by A. O. Barut and B. Kurşunoglu, Proceedings of the Conference on Symmetry Principles at High Energy, Coral Gables, Florida, January 1964 (to be published); also Phys. Rev. (to be published).

[2] H. Fröhlich, Nucl. Phys. 45, 609 (1963).

[3] W. D. McGlinn, Phys. Rev. Letters 12, 467 (1964).

[4] Throughout this paper the term "product" of two Lie groups (or sum of the Lie algebras) will denote a product of the groups (sum of the algebras) for which the generators of each of the two components maintain their original commutation relations, but do not commute with each other, the commutator being linear in the generators of *both* groups. If the commutator is linear only in the generators of one of the components, that group is an invariant subgroup and the product becomes a semidirect product. For a definition of semidirect products see, e.g., E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* (Academic Press Inc., New York, 1963), Vol. I, p. 6. The following notations will be used: P_μ, $M_{\mu\nu}$ (μ, $\nu=0, 1, 2, 3$) are the generators of the inhomogeneous Lorentz group (Poincaré Group), X_ρ are the generators of the internal symmetry group, $[X_\rho,X_\sigma]=\sum_\gamma c_{\rho\sigma}{}^\gamma X_\gamma$. For simple algebras we use the Cartan-Weyl basis: the commuting generators will be denoted by H_l, the "ladder" generators by E_α:

$$[H_l,H_k]=0 \quad l, k=1, 2, \cdots, r \ (r=\text{rank}),$$
$$[E_\alpha,E_{-\alpha}]=\sum_l r_l(\alpha)H_l,$$
$$[H_l,E_\alpha]=r_l(\alpha)E_\alpha,$$
$$[E_\alpha,E_\beta]=N_{\alpha\beta}E_{\alpha+\beta} \quad (\alpha\neq-\beta),$$

[cf. e.g., R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. 34, 1 (1962)].

II. LORENTZ GROUP AND SYMMETRIES

Theorem 1. [*McGlinn (Ref. 3).*] Let X_ρ be the set of generators of the internal symmetry group S and P_μ, $M_{\mu\nu}$ the generators of the (proper orthochronous inhomogeneous) Lorentz group $\mathfrak{L}$. If the set of these generators is closed under commutation and if

$$[X_\rho,M_{\mu\nu}]=0, \tag{1}$$

then

$$[[X_\rho,X_\sigma],P_\mu]=0. \tag{2}$$

If S has no Abelian factor groups we can further show that

$$[X_\rho,P_\mu]=0. \tag{3}$$

Proof: Let

$$[X_\rho,P_\mu]=\sum_\sigma a_{\rho\mu\sigma}X_\sigma+\sum_\nu b_{\rho\mu\nu}P_\nu+\sum_{(\nu\lambda)} c_{\rho\mu(\nu\lambda)}M_{\nu\lambda}. \tag{4}$$

Taking the commutator of both sides with $M_{\lambda\mu}$ we may deduce

$$a_{\rho\mu\sigma}=0; \quad b_{\rho\mu\nu}=b(\rho)\delta_{\mu\nu}; \quad c_{\rho\mu(\nu\lambda)}=0. \tag{5}$$

We thus have

$$[X_\rho,P_\mu]=b(\rho)P_\mu. \tag{6}$$

Hence

$$[[X_\rho,X_\sigma],P_\mu]=0. \tag{7}$$

Since $[[X_\rho,X_\sigma],P_\mu]=0$, it is clear that if every element X_ρ can be expressed as a linear combination of commutators $[X_\sigma,X_\tau]$, then $[X_\rho,P_\mu]=0$. If, on the other hand, there exist elements which cannot be spanned by the commutators, we cannot then deduce this result; but in such a case the group generated by the commutators of the generators of S is an invariant subgroup of S and the factor group S/S_1 is Abelian. Conversely, by a theorem stated in Pontrjagin,[5] we can deduce that if there exist (nontrivial) Abelian factor groups S/S_2, then S_1 is a subgroup of S_2 and consequently the commutators do not span the generators of S. We note also that given such a situation we could always choose to redefine the elements not spanned by the commutators such that at most one generator fails to commute with the Hamiltonian.

These results suggest that in any scheme of incorporating internal symmetries of a conventional kind (simple groups!) into a large group subsuming the Lorentz group we may have to give up the postulate of commutation of the homogeneous Lorentz generators with the generators of S. As a matter of fact we may drop the requirement that all generators X_ρ commute with the $M_{\mu\nu}$ but restrict it to only the complete commuting set of generators (the additive quantum numbers) and require them to commute with all the generators of the Lorentz group: we would like the strong interactions, even with mass splitting, to conserve both the electric charge and the hypercharge. The following theorem states that no mass splittings can be obtained within this framework either.

Theorem 2.[6] Let S be a simple symmetry group with the generators H_l, E_α in the Weyl basis and let us consider the product of the symmetry group S and the Lorentz group satisfying the Lorentz invariance of H_l

$$[H_l,P_\mu]=0, \quad [H_l,M_{\mu\nu}]=0. \tag{8}$$

Then

$$[E_\alpha,P_\mu]=0; \quad [E_\alpha,M_{\mu\nu}]=0. \tag{9}$$

Proof: Let us write

$$[P_\mu,E_\alpha]=\sum_\beta a_{\mu\alpha\beta}E_\beta+\sum_l b_{\mu\alpha l}H_l +\sum_\nu c_{\mu\alpha\nu}P_\nu+\sum_{(\nu\sigma)} d_{\mu\alpha(\nu\sigma)}M_{\nu\sigma}. \tag{10}$$

Since $[H_m,[P_\mu,E_\alpha]]=r_m(\alpha)[P_\mu,E_\alpha]$, it follows that

$$\sum_\beta a_{\mu\alpha\beta}\{r_m(\alpha)-r_m(\beta)\}E_\beta+\sum_l r_m(\alpha)b_{\mu\alpha l}H_l +\sum_\nu r_m(\alpha)c_{\mu\alpha\nu}P_\nu+\sum_{(\nu\sigma)} r_m(\alpha)d_{\mu\alpha(\nu\sigma)}M_{\nu\sigma}=0. \tag{11}$$

Comparing coefficients we get

$$b_{\mu\alpha l}=0; \quad c_{\mu\alpha\nu}=0; \quad d_{\mu\alpha(\nu\sigma)}=0, \quad a_{\mu\alpha\beta}=0 \quad \text{for} \quad \alpha\neq\beta. \tag{12}$$

Hence we may write

$$[P_\mu,E_\alpha]=a(\mu\alpha)E_\alpha. \tag{13}$$

Similarly,

$$[M_{\nu\sigma},E_\alpha]=a((\nu\sigma)\alpha)E_\alpha. \tag{14}$$

Hence

$$[[M_{\nu\sigma},P_\mu],E_\alpha]=0, \tag{15}$$

$$[[M_{\nu\sigma},M_{\lambda\mu}],E_\alpha]=0. \tag{16}$$

Hence it follows that

$$[P_\mu,E_\alpha]=0, \quad [M_{\nu\sigma},E_\alpha]=0, \tag{17}$$

which was the assertion.

We remark here that in view of the commutation relations of the generators of the Lorentz group, we can replace the commutability of H_l with P_μ, $M_{\mu\nu}$ by the (trivially weaker) requirement of commutability of H_l with P_0, $M_{0\nu}$ only. A related comment applies to the previous theorem.

Theorem 3. If in the product of the Lorentz group and the (simple) internal symmetry group S we have the relations

$$[M_{0\nu},E_\alpha]=0, \tag{18}$$

then the product simplifies to a direct product.

Proof: From the remark above we deduce immediately

$$[M_{\mu\nu},E_\alpha]=0, \tag{19}$$

so that

$$[M_{\mu\nu},[E_\alpha,E_{-\alpha}]]=0. \tag{20}$$

But

$$[E_\alpha,E_{-\alpha}]=\sum_l r_l(\alpha)H_l \tag{21}$$

so that

$$\sum_l r_l(\alpha)[M_{\mu\nu},H_l]=0. \tag{22}$$

But since the vectors $r_l(\alpha)$ span the l-dimensional root vector space this implies

$$[M_{\mu\nu},H_l]=0. \tag{23}$$

We have now shown that

$$[M_{\mu\nu},E_\alpha]=0; \quad [M_{\mu\nu},H_l]=0; \tag{24}$$

and hence, by McGlinn's theorem, we deduce that

$$[P_\mu,E_\alpha]=0; \quad [P_\mu,H_l]=0. \tag{25}$$

Let us now consider the case where S is an invariant subalgebra of the Lie algebra associated with the product of the groups S and $\mathfrak{L}$. We may then consider $\mathfrak{L}$ as a set of outer automorphisms of the algebra S and the product of $\mathfrak{L}$ and S becomes a *semidirect* product[4]:

Theorem 4: If in the semidirect product of the inhomogeneous Lorentz group $\mathfrak{L}$ and the (simple) internal

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[5] L. S. Pontrjagin, *Topological Groups* (Moscow, 1954), 2nd (Russian) ed., Chap. X; [English transl.: (Princeton University Press, Princeton, New Jersey, 1939), Chap. IX].

[6] This theorem has also been proved independently by F. Coester, M. Hamermesh and W. D. McGlinn, Phys. Rev. 135, B451 (1964). Other work along similar lines has been done by O. W. Greenberg, Phys. Rev. 135, B1447 (1964). We thank these authors for prepublication copies of their work.

symmetry group S, we have the relations

$$[H_l,M_{\mu\nu}]=0, \tag{26}$$

then

$$[H_l,P_\mu]=[E_\alpha,M_{\mu\nu}]=[E_\alpha,P_\mu]=0, \tag{27}$$

and the semidirect product reduces to a direct product.

Proof: We may write

$$[M_{\mu\nu},E_\alpha]=\sum_l a_{(\mu\nu)\alpha l}H_l+\sum_\beta b_{(\mu\nu)\alpha\beta}E_\beta. \tag{28}$$

Then

$$\begin{aligned}[H_m,[M_{\mu\nu},E_\alpha]] &=\sum_\beta b_{(\mu\nu)\alpha\beta}r_m(\beta)E_\beta \\ &=r_m(\alpha)[M_{\mu\nu},E_\alpha] \\ &=r_m(\alpha)\{\sum_l a_{(\mu\nu)\alpha l}H_l+\sum_\beta b_{(\mu\nu)\alpha\beta}E_\beta\}.\end{aligned} \tag{29}$$

Hence

$$r_m(\alpha)a_{(\mu\nu)\alpha l}=0 \quad \text{and} \quad \{r_m(\alpha)-r_m(\beta)\}b_{(\mu\nu)\alpha\beta}=0,$$

so that

$$a_{(\mu\nu)\alpha l}=0 \quad \text{and} \quad b_{(\mu\nu)\alpha\beta}=b_{(\mu\nu)\alpha}\delta_{\alpha\beta}.$$

Thus

$$[M_{\mu\nu},E_\alpha]=b_{(\mu\nu)\alpha}E_\alpha. \tag{30}$$

But then

$$[[M_{\lambda\sigma},M_{\mu\nu}],E_\alpha]=0, \tag{31}$$

and this implies

$$[M_{\mu\nu},E_\alpha]=0, \tag{32}$$

so that, by McGlinn's theorem, the full group is a direct product.

We may also consider the rather "unphysical" case where the roles of $\mathfrak{L}$ and S in the preceding theorem are interchanged. Indeed, whereas under the assumptions of Theorem 4 (to be denoted briefly by $[\mathfrak{L},S]\subset S$, i.e., that commutators of elements in $\mathfrak{L}$ and S are linear in S) one could conceivably accept the fact that $\mathfrak{L}$ is an outer automorphism of S—i.e., "motions" of particles affect their "internal" properties (e.g., isospin projections)—the reversed situation ($[\mathfrak{L},S]\subset\mathfrak{L}$) where S is an automorphism of $\mathfrak{L}$ ("internal" transformations affect energy-momentum) is apparently devoid of any physical meaning.

Nevertheless, we were able to prove that the preceding results can be established under these circumstances also. In view of the limited physical interest of these assumptions, we state the appropriate theorem without giving the details of the proof:

Theorem 5. Theorem 4 remains true if we replace the requirement $[\mathfrak{L},S]\subset S$ by the requirement $[\mathfrak{L},S]\subset\mathfrak{L}$, i.e., assuming

$$[H_l,M_{\mu\nu}]=0 \tag{33}$$

and

$$[H_l,P_\mu]=\sum_\sigma e_{l\mu\sigma}P_\sigma+\sum_{(\sigma\tau)} f_{l\mu(\sigma\tau)}M_{\sigma\tau}, \tag{34}$$

one can prove that

$$[E_\alpha,M_{\mu\nu}]=0, \tag{35}$$

and hence by McGlinn's theorem, also

$$[H_l,P_\mu]=0, \tag{36}$$

$$[E_\alpha,P_\mu]=0. \tag{37}$$

III. GALILEI GROUP AND SYMMETRIES

In the above demonstrations we have made explicit use of the commutation relations of the Lorentz group. It would be interesting to see if we could get *qualitatively* different results if we replace the Lorentz group by the Galilei group. We find that this is indeed the case.

Our notation is as follows: H is the generator of time translations, P_j are the generators of space translations, G_j are the generators of velocity along the j axis, and M_{jk} are the generators of rotation in the j,k plane.

Theorem 6. Consider the product of the Galilei group and an arbitrary simple (internal symmetry) group with generators H_l, E_α. If

$$[H_l,P_j]=[H_l,G_j]=[H_l,M_{jk}]=0, \tag{38}$$

then

$$[E_\alpha,P_j]=[E_\alpha,G_j]=[E_\alpha,M_{jk}]=0 \tag{39}$$

and there exists a set of numbers $p(l)$ and $q(\alpha)$ such that

$$\begin{aligned}&[H-\sum_l p(l)H_l-\sum_\alpha q(\alpha)E_\alpha,\ H_m]=0,\\ &[H-\sum_l p(l)H_l-\sum_\alpha q(\alpha)E_\alpha,\ E_\beta]=0.\end{aligned} \tag{40}$$

Proof: Let

$$[H_l,H]=a_lH+\sum_j b_{lj}P_j+\sum_j c_{lj}G_j+\sum_{(jk)} d_{l(jk)}M_{jk} +\sum_m e_{lm}H_m+\sum_\alpha f_{l\alpha}E_\alpha. \tag{41}$$

Since

$$[M_{jk},H_l]=[M_{jk},H]=0 \tag{42}$$

by evaluating $[M_{jk},[H_l,H]]$ it follows that

$$b_{lj}=c_{lj}=d_{l(jk)}=0. \tag{43}$$

Hence

$$[H_l,H]=a_lH+\sum_m e_{lm}H_m+\sum_\alpha f_{l\alpha}E_\alpha. \tag{44}$$

Let us write

$$[P_j,E_\alpha]=x_{j\alpha}H+\sum_l y_{j\alpha l}H_l+\sum_\beta z_{j\alpha\beta}E_\beta +\sum_k u_{j\alpha k}P_k+\sum_k v_{j\alpha k}G_k+\sum_{(kl)} w_{j\alpha(kl)}M_{kl}. \tag{45}$$

Using the result

$$[H_n,[P_j,E_\alpha]]=r_n(\alpha)[P_j,E_\alpha], \tag{46}$$

we can deduce

$$u_{j\alpha k}=v_{j\alpha k}=w_{j\alpha(kl)}=0 \tag{47}$$

as well as the relations

$$x_{j\alpha}\{a_l-r_n(\alpha)\}=0,\quad y_{j\alpha m}r_n(\alpha)-x_{j\alpha}e_{nm}=0,\quad \{r_n(\alpha)-r_n(\beta)\}z_{j\alpha\beta}=0. \tag{48}$$

But since

$$0=[G_j,[H_l,H]]=a_lP_j+\sum_\alpha f_{l\alpha}[G_j,E_\alpha] \tag{49}$$

and

$$[H_m,[G_j,E_\alpha]]=r_m(\alpha)[G_j,E_\alpha], \tag{50}$$

it follows that

$$a_l=0. \tag{51}$$

Using this in the above equations and making use of the properties of the root vectors, we obtain

$$x_{j\alpha}=0,\quad y_{j\alpha m}=0,\quad z_{j\alpha\beta}=\xi(j\alpha)\delta_{\alpha\beta}. \tag{52}$$

Thus

$$[H_l,H]=\sum_m e_{lm}H_m+\sum_\alpha f_{l\alpha}E_\alpha,\quad [P_j,E_\alpha]=\xi(j\alpha)E_\alpha. \tag{53}$$

Using

$$[H_n,[G_j,E_\alpha]]=r_n(\alpha)[G_j,E_\alpha],\quad [H_n,[M_{jk},E_\alpha]]=r_n(\alpha)[M_{jk},E_\alpha], \tag{54}$$

we may deduce in a similar fashion

$$[G_j,E_\alpha]=\eta(j\alpha)E_\alpha,\quad [M_{jk},E_\alpha]=\zeta(jk\alpha)E_\alpha. \tag{55}$$

But this implies that

$$[[P_i,M_{jk}],E_\alpha]=0,\quad [[G_i,M_{jk}],E_\alpha]=0,\quad [[M_{ij},M_{kl}],E_\alpha]=0. \tag{56}$$

Consequently,

$$[P_j,E_\alpha]=[G_j,E_\alpha]=[M_{jk},E_\alpha]=0. \tag{57}$$

We now evaluate

$$[H_l,[H_m,H]]=[H_l,\sum_n e_{mn}H_n+\sum_\alpha f_{m\alpha}E_\alpha]=\sum_\alpha f_{m\alpha}r_l(\alpha)E_\alpha. \tag{58}$$

Since the left-hand side is symmetric in l and m it follows that

$$r_l(\alpha)f_{m\alpha}=r_m(\alpha)f_{l\alpha}, \tag{59}$$

which requires that

$$f_{l\alpha}=q(\alpha)r_l(\alpha). \tag{60}$$

Now consider the quantity

$$\epsilon_1=H-\sum_\alpha q(\alpha)E_\alpha. \tag{61}$$

Then

$$[H_l,\epsilon_1]=\sum_m e_{lm}H_m,\quad [P_j,\epsilon_1]=[M_{jk},\epsilon_1]=0,\quad [G_j,\epsilon_1]=[G_j,H]=P_j. \tag{62}$$

Since

$$[P_j,[\epsilon_1,E_\alpha]]=[G_j,[\epsilon_1,E_\alpha]]=[M_{jk},[\epsilon_1,E_\alpha]], \tag{63}$$

we may write (without any loss of generality)

$$[\epsilon_1,E_\alpha]=\sum_l g_{\alpha l}H_l+\sum_\beta h_{\alpha\beta}E_\beta. \tag{64}$$

Then

$$r_n(\alpha)\{\sum_l g_{\alpha l}H_l+\sum_\beta h_{\alpha\beta}E_\beta\}=[H_n,[\epsilon_1,E_\alpha]]=\sum_\beta h_{\alpha\beta}r_n(\beta)E_\beta. \tag{65}$$

Hence

$$g_{\alpha l}=0,\quad h_{\alpha\beta}=h(\alpha)\delta_{\alpha\beta}, \tag{66}$$

so that

$$[\epsilon_1,E_\alpha]=h(\alpha)E_\alpha. \tag{67}$$

But since

$$[E_\alpha,E_\beta]=N_{\alpha\beta}E_{\alpha+\beta},\quad \alpha\neq-\beta, \tag{68}$$

we have the relation

$$h(\alpha)+h(\beta)=h(\alpha+\beta) \tag{69}$$

[which is valid for $\alpha=-\beta$ also if we define $h(0)=0$]. But any such quantity $h(\alpha)$ may be expressed as a linear sum

$$h(\alpha)=\sum_l p(l)r_l(\alpha). \tag{70}$$

Now if we define

$$\epsilon_0=\epsilon_1-\sum_l p(l)H_l=H-\sum_l p(l)H_l-\sum_\alpha q(\alpha)E_\alpha, \tag{71}$$

we may deduce

$$[\epsilon_0,H_l]=[\epsilon_0,E_\alpha]=0. \tag{72}$$

The results obtained so far are the best possible: we cannot deduce anything better. To see this we consider a direct product of the Galilei group and the internal symmetry group. Then we may make the transcription

$$H\to H+\sum_l p(l)H_l+\sum_\alpha q(\alpha)E_\alpha \tag{73}$$

without affecting the Galilei group commutation relation or the hypothesis.

It is interesting to see that the "broken symmetry" so obtained has the property that the Hamiltonian transforms as the sum of an invariant and a linear combination of the generators of the internal symmetry group; by a redefinition of the basic elements of the internal symmetry algebra we can make the term noninvariant with respect to the internal symmetry group to be proportional to a single H_l. We have thus a situation familiar from the phenomenological theories of Gell-Mann and Okubo.

The above theorem is the Galilei analog of Theorem 2. We may now state the analog of McGlinn's theorem.

Theorem 7. Consider a product of the Galilei group and a simple internal symmetry group. If

$$[H_l,M_{jk}]=[H_l,G_j]=0,\quad [M_{jk},E_\alpha]=0,\tag{74}$$

then

$$[H_l,P_j]=0,\quad [P_j,E_\alpha]=[G_j,E_\alpha]=0,\tag{75}$$

and there exist numbers $p(l)$, $q(\alpha)$ such that

$$\begin{aligned}&[H-\sum_l p(l)H_l-\sum_\alpha q(\alpha)E_\alpha,\ H_m]=0,\\ &[H-\sum_l p(l)H_l-\sum_\alpha q(\alpha)E_\alpha,\ E_\beta]=0.\end{aligned}\tag{76}$$

Proof: As in the proof of the previous theorem, we can write

$$[H_l,H]=a_lH+\sum_m e_{lm}H_m+\sum_\alpha f_{l\alpha}E_\alpha.\tag{77}$$

Taking the commutator of both sides with respect to G_j we get

$$[H_l,P_j]=a_lP_j+\sum_\alpha f_{l\alpha}[G_j,E_\alpha].\tag{78}$$

Since $[M_{jk},E_\alpha]=0$, we may write without loss of generality,

$$[P_j,E_\alpha]=b_\alpha P_j+c_\alpha G_j+\tfrac{1}{2}d_\alpha\sum_{(kl)}\epsilon_{jkl}M_{kl},\tag{79}$$

where ϵ_{jkl} is the Levi-Civita symbol. Hence

$$[P_j,[P_k,E_\alpha]]=\tfrac{1}{2}d_\alpha\sum_{(lm)}\epsilon_{klm}[P_j,M_{lm}].\tag{80}$$

But the left-hand side is symmetric in j and k. Hence

$$\tfrac{1}{2}d_\alpha\sum_{(lm)}[\epsilon_{klm}P_j-\epsilon_{jlm}P_k,\ M_{lm}]=0,\tag{81}$$

so that $d_\alpha=0$ and hence,

$$[P_j,E_\alpha]=b_\alpha P_j+c_\alpha G_j.\tag{82}$$

In exactly similar fashion we may write

$$[G_j,E_\alpha]=g_\alpha P_j+h_\alpha G_j.\tag{83}$$

But then, since

$$[H_l,[G_j,E_\alpha]]=r_l(\alpha)[G_j,E_\alpha]=0,\tag{84}$$

it follows that

$$[G_j,E_\alpha]=0.\tag{85}$$

Using this we can deduce that

$$[H_l,P_j]=a_lP_j.\tag{86}$$

Now

$$\begin{aligned}[H_l,[P_j,E_\alpha]]&=b_\alpha[H_l,P_j]=b_\alpha a_lP_j\\ &=[[H_l,P_j],E_\alpha]-[[H_l,E_\alpha],P_j]\\ &=\{a_l+r_l(\alpha)\}\{b_\alpha P_j+c_\alpha G_j\}.\end{aligned}\tag{87}$$

Hence

$$r_l(\alpha)b_\alpha=0,\tag{88}$$

so that

$$b_\alpha=0,\quad [P_j,E_\alpha]=c_\alpha G_j.\tag{89}$$

On the other hand, using the commutation properties of M_{kl} we can write

$$[H,E_\alpha]=u_\alpha H+\sum_l v_{\alpha l}H_l+\sum_\beta w_{\alpha\beta}E_\beta,\tag{90}$$

so that

$$[[G_j,H],E_\alpha]=u_\alpha[G_j,H].\tag{91}$$

Comparing this with $[P_j,E_\alpha]=c_\alpha G_j$, we deduce that

$$c_\alpha=0,\quad [P_j,E_\alpha]=0.\tag{92}$$

We now have

$$[P_j,[E_\alpha,E_{-\alpha}]]=\sum_l r_{-l}(\alpha)[P_j,H_l]=0,\tag{93}$$

so that

$$[P_j,H_l]=0.\tag{94}$$

We may now use Theorem 6 to complete the proof of Theorem 7.

IV. DISCUSSION

It is not very surprising that we could get "broken" symmetry with the Galilei group, but it is surprising that the only kind of breakdown in the symmetry is by the addition of a term *linear* in the generators of the internal symmetry group to an otherwise invariant Hamiltonian. In addition, the coefficients of these generators and the corresponding coefficients of the "mass formula" are independent of the representation. It is equally surprising that no such possibility obtains for the case of the Lorentz group.

If we want a breakdown manifested by an effective Hamiltonian containing nonlinear terms in the generators (like the Gell-Mann–Okubo formula) within the present framework, we must work with an internal-symmetry algebra containing these elements; we must adjoint elements like Y^2, $I(I+1)$, etc., to the algebra. With these modifications we are essentially working with a higher symmetry group for each multiplet and could obtain in principle the higher order mass formulas of Okubo as well. It is interesting to note that the linear mass formulas are fair approximations to the empirical situation.

We might also remark on the possibility of relativistic theories with mass splittings within multiplets. Such theories can certainly be constructed (at least as far as any relativistic theory may be constructed!) but Theorem 2 then demands that the commutators of the elements of the simple internal symmetry algebra and the Lorentz group generators cannot be expanded as linear combinations of the two sets of generators.